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Reg. No.....

Name.....

M.Sc. DEGREE (C.S.S.) EXAMINATION, MARCH 2022

First Semester

Faculty of Science

Branch I (a)—Mathematics

MT 01 C01—LINEAR ALGEBRA

(2012–2018 Supplementary/Mercy Chance)

Time : Three Hours

Maximum Weight : 30

Part A

Answer any **five** *questions.*
Each question carries weight 1.

1. Are the vectors (8, 1, – 2, 4), (2, 1, 8, – 3), (1, 7, – 6, 0), (4, 7, 1, – 6), linearly independent in $\mathbb{R}^4$.
2. Prove that the space of all $m \times n$ matrices over the field F has dimension mn by exhibiting a basis.
3. Consider $f : F^2 \to F$ defined by $f(x_1, x_2) = ax_1 + bx_2$ is a linear functional. Find $T^t f$ if $T(x_1, x_2) = (-x_2, x_1)$.
4. Find the dual of the basis B : = {(1, 0, – 1), (1, 1, 1), (2, 2, 0)} for $\mathbb{C}^3$.
5. Let A be a 2 × 2 matrix over a field F. Prove that det (I + A) = 1 + det (A) if and only if trace (A) = 0.
6. Let K = R [x], the ring of polynomials over the field of real numbers. Find the determinant of $B = \begin{bmatrix} x^2 - 1 & x + 2 \\ x^2 - 2x + 3 & x \end{bmatrix}$.
7. Consider the space of all continuous real-valued functions on the real line. Show that the linear operator $(Tf)(x) = \int_0^x f(t)\,dt$ has no eigenvalues.
8. Find the minimal polynomial of the matrix $\begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 4 \\ 0 & 1 & 7 \end{bmatrix}$.

(5 × 1 = 5)

Turn over

Part B

Answer any **five** *questions.*
Each question carries weight 2.

9. In $\mathbb{C}^3$, show that $\{(1, 0, -i), (1+i, 1-i, 1), (i, i, i)\}$ form a basis, and find the co-ordinates of the vector (a, b, c) in this basis.

10. Let W_1 and W_2 are subspaces of a finite-dimensional vector space V. Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

11. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$. Find the matrix of T in the ordered basis $\{(1, 0, 1), (-1, 2, 1), (2, 1, 1)\}$.

12. If W is a k-dimensional subspace of an n-dimensional vector space V, then prove that W is the intersection of $(n-k)$ hyperspaces in V.

13. Prove that the determinant of a triangular matrix is the product of its diagonal entries.

14. Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$. Prove that T is diagonalizable.

15. Suppose that $f = (x - c_1)^{d_1} \cdots (x - c_k)^{d_k}$ is the characteristic polynomial of an $n \times n$ matrix A. Show that trace $A = c_1 d_1 + c_2 d_2 + \ldots + c_k d_k$.

16. Suppose A, B, C and D are $n \times n$ complex matrices that commute. Show that

$$\det \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \det (AD - BC).$$

$(5 \times 2 = 10)$

Part C

Answer any three *questions.*
Each question carries weight 5.

17. Show that Row-equivalent matrices have the same row space.

18. Consider the linear operator T $(x_1, x_2) = (-x_2, x_1)$, prove that (T – cI) is invertible for every real number c.

19. Let K be a commutative ring with identity and let n be a positive integer. Show that there is precisely one determinant function on the set of $n \times n$ matrices over K.

20. If A and B are $n \times n$ complex matrices, show that AB — BA = I is not possible.

21. Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Prove that T is diagonalizable if and only if the minimal polynomial for T is of the form $p = (x - c_1) \ldots (x - c_k)$, where $c_1, \ldots, c_k$ are distinct elements of F.

22. Suppose $\mathcal{J}$ is a commuting family of diagonalizable linear operators on the finite dimensional vector space V. Show that there exists an ordered basis for V such that every operator in $\mathcal{J}$ is represented in that basis by a diagonal matrix.

$(3 \times 5 = 15)$